

Some Generalizations of the MacMahon Master Theorem

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Abstract

We consider a number of generalizations of the β -extended MacMahon Master Theorem for a matrix. The generalizations are based on replacing permutations on multisets formed from matrix indices by partial permutations or derangements over matrix or submatrix indices.

1 Introduction

The Master Theorem due to Percy MacMahon first appeared in 1915 in his classic text *Combinatory Analysis* [MM]. A generalization known as the β -extended MacMahon Master Theorem was discovered in more recent times by Foata and Zeilberger [FZ]. This present paper is concerned with several further generalizations of the β -extension informed by recent results in the theory of vertex operator algebras concerning the partition and correlation functions on a genus zero and higher Riemann surface [MT1, MT2, TZ1, HT, TZ2, TZ3].

One formulation of the MacMahon Master Theorem (MMT) is the identity of $\det(I - A)^{-1}$, for a given matrix A , to an infinite weighted sum over all permanents for matrices indexed by multisets formed from the indices of A [W, KP]. The β -extended MMT relates $\det(I - A)^{-\beta}$ to a similar sum

over so-called β -extended permanents [FZ, KP]. We consider the following generalizations:

- (i) The Submatrix MMT. Here the infinite sum runs over multisets formed from the indices of a given submatrix of A .
- (ii) The Partial Permutation MMT. In this case the β -extended permanent is replaced by what we refer to as a (β, θ, ϕ) -extended partial permanent defined in terms of a sum over all partial permutations of the A -indices.
- (iii) The Derangement MMT. We replace the β -extended permanent by what we refer to as a β -extended deranged partial permanent defined in terms of a sum over the derangements of the A -indices.

We begin in Section 2 with a review of the β -extended MMT [FZ]. We provide a graph theoretic proof based on an enumeration of appropriate weights of non-isomorphic permutation graphs labelled by multisets of the indexing set for A . In particular, the connected subgraphs are cycles corresponding to permutation cycles. Section 3 describes our first generalization, the Submatrix MMT (Theorem 3.1), where the set of permutation graphs is modified to account for multisets formed from the indices of an A submatrix. In Section 4 we introduce the (β, θ, ϕ) -extended partial permanent of a matrix, a variation on the β -extended permanent involving a sum over the partial permutations of the matrix indices. The corresponding Partial Permutation MMT (Theorem 4.1) is proved by a consideration of partial permutation graphs whose connected subgraphs are cycles and open necklaces. Section 5 combines both of the previous generalizations into one general result in Theorem 5.1. Finally, in Section 6 we introduce another variation, the β -extended deranged permanent of a matrix, where we sum over the derangements (fixed point free permutations) of the matrix indices. We conclude with a Derangement MMT (Theorem 6.1) and a corresponding Submatrix Derangement MMT (Theorem 6.2) which are proved by applying the graph theory description of Sections 2 and 3 respectively, but where no 1-cycle graphs occur.

2 The β -Extended MacMahon Master Theorem

Let $A = (A_{ij})$ be an $n \times n$ matrix indexed by $i, j \in \{1, \dots, n\}$. The β -extended Permanent of A is defined by [FZ], [KP]

$$\text{perm}_\beta A = \sum_{\pi \in \Sigma_n} \beta^{C(\pi)} \prod_{i=1}^n A_{i\pi(i)}, \quad (1)$$

where $C(\pi)$ is the number of cycles in $\pi \in \Sigma_n$, the symmetric group. The permanent and determinant are the special cases:

$$\text{perm } A = \text{perm}_{+1} A, \quad \det(-A) = \text{perm}_{-1} A. \quad (2)$$

Let $\mathbf{r} = (r_1, \dots, r_n)$ denote an n -tuple of non-negative integers. Define

$$\mathbf{r}! = r_1! \dots r_n!, \quad (3)$$

and let

$$n^{\mathbf{r}} = \{1^{r_1} 2^{r_2} \dots n^{r_n}\} = \{1_1, \dots, 1_{r_1}, \dots, n_1, \dots, n_{r_n}\}, \quad (4)$$

denote the multiset of size $N = \sum_{i=1}^n r_i$ formed from the original index set $\{1, \dots, n\}$ where the index i is repeated r_i times. We sometimes notate a repeated index by i_a for label $a = 1, \dots, r_i$. For an $n \times n$ matrix A , we let $A(n^{\mathbf{r}}, n^{\mathbf{r}})$ denote the $N \times N$ matrix indexed by the elements of $n^{\mathbf{r}}$ and define $A(n^{\mathbf{r}}, n^{\mathbf{r}}) = 1$ for $\mathbf{r} = (0, 0, \dots, 0)$.

We now describe a generalization, due to Foata and Zeilberger [FZ], of the MacMahon Master Theorem (MMT) of classical combinatorics [MM]. We give a detailed proof based on a graph theory method which is extensively employed throughout this paper. This proof is very similar to that of Theorem 5 of [MT1] where the MMT was essentially rediscovered.

Theorem 2.1 (The β -Extended MMT)

$$\sum_{r_i \geq 0} \frac{1}{\mathbf{r}!} \text{perm}_\beta A(n^{\mathbf{r}}, n^{\mathbf{r}}) = \frac{1}{\det(I - A)^\beta}. \quad (5)$$

Remark 2.2 For $\beta = 1$, Theorem 2.1 reduces to the MMT. For $\beta = -1$ we use (2) to find that only proper subsets of $\{1, \dots, n\}$ contribute resulting in the determinant identity for $B = -A$ e.g. [TZ1]

$$\sum_{r_i \in \{0,1\}} \det B(n^{\mathbf{r}}, n^{\mathbf{r}}) = \det(I + B).$$

Proof of Theorem 2.1. Let $\Sigma(n^{\mathbf{r}})$ denote the symmetric group of the multiset $n^{\mathbf{r}}$. For $\pi \in \Sigma(n^{\mathbf{r}})$ we define a permutation graph γ_π with N vertices labelled by $i \in \{1, \dots, n\}$, and with directed edges

$$e_{ij} = i \bullet \longrightarrow \bullet j,$$

provided $j = \pi(i)$. The connected subgraphs of $\gamma_\pi \in \Gamma$ are cycles arising from the cycles of π . For example, for $n = 4$ with $\mathbf{r} = (3, 2, 0, 1)$ and permutation $\pi = (1_1 2_1 1_2 2_2)(1_3 4_1)$ the corresponding graph has two cycles as shown in Fig. 1

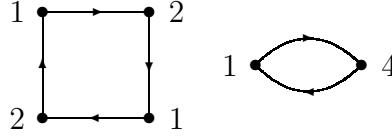


Fig. 1 γ_π for $\pi = (1_1 2_1 1_2 2_2)(1_3 4_1)$.

Define a weight for each edge of γ_π by

$$w(e_{ij}) = A_{ij},$$

and a weight for γ_π by

$$w(\gamma_\pi) = \beta^{C(\pi)} \prod_{e_{ij} \in \gamma_\pi} w(e_{ij}). \quad (6)$$

where $C(\pi)$ is the number of cycles in π . Note that the weight is multiplicative with respect to the cycle decomposition of π . (6) also implies

$$\text{perm}_\beta A(n^{\mathbf{r}}, n^{\mathbf{r}}) = \sum_{\pi \in \Sigma(n^{\mathbf{r}})} w(\gamma_\pi). \quad (7)$$

Let $\Lambda(\mathbf{r}) = \Sigma_{r_1} \times \dots \times \Sigma_{r_n} \subseteq \Sigma(n^{\mathbf{r}})$ denote the label group of order $|\Lambda(\mathbf{r})| = \mathbf{r}!$ which permutes the identical elements of $n^{\mathbf{r}}$. $\Lambda(\mathbf{r})$ generates

isomorphic graphs with $\gamma_\pi \sim \gamma_{\lambda\pi\lambda^{-1}}$ for $\lambda \in \Lambda(\mathbf{r})$ and the automorphism group of γ_π is the π stabilizer $\text{Aut}(\gamma_\pi) = \{\lambda \in \Lambda(\mathbf{r}) | \lambda\pi = \pi\lambda\} \subseteq \Lambda(\mathbf{r})$. Using the Orbit-Stabilizer theorem it follows that the number of isomorphic graphs generated by the action of $\Lambda(\mathbf{r})$ on γ_π is given by

$$|\Lambda(\mathbf{r})\gamma_\pi| = \frac{|\Lambda(\mathbf{r})|}{|\text{Aut}(\gamma_\pi)|}. \quad (8)$$

(e.g. in Fig. 1, $\Lambda(\mathbf{r}) = \Sigma_2 \times \Sigma_3$ and $\text{Aut}(\gamma_\pi) = \Sigma_2$ so that there are 6 permutations in $\Sigma(n^{\mathbf{r}})$ with graph γ_π). Combining (7) and (8) we find that

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \text{perm}_\beta A(n^{\mathbf{r}}, n^{\mathbf{r}}) = \sum_{\gamma \in \Gamma} \frac{w(\gamma)}{|\text{Aut}(\gamma)|}, \quad (9)$$

where Γ denotes the set of non-isomorphic graphs.

Consider the decomposition of a graph γ into cycle graphs

$$\gamma = \gamma_{\sigma_1}^{m_1} \cdots \gamma_{\sigma_K}^{m_K},$$

where $\{\gamma_{\sigma_i}\}$ are non-isomorphic and γ_{σ_i} occurs m_i times. The automorphism group is

$$\text{Aut}(\gamma) = \prod_{i=1}^M \text{Aut}(\gamma_{\sigma_i}^{m_i}),$$

where $\text{Aut}(\gamma_\sigma^m) = \Sigma_m \rtimes \text{Aut}(\gamma_\sigma)^m$ of order $m! |\text{Aut}(\gamma_\sigma)|^m$. Furthermore, since the weight is multiplicative, $w(\gamma) = \prod_{i=1}^M w(\gamma_{\sigma_i})^{m_i}$. Thus we find

$$\begin{aligned} \sum_{\gamma \in \Gamma} \frac{w(\gamma)}{|\text{Aut}(\gamma)|} &= \prod_{\gamma_\sigma \in \Gamma_\sigma} \sum_{m \geq 0} \frac{1}{m!} \left(\frac{w(\gamma_\sigma)}{|\text{Aut}(\gamma_\sigma)|} \right)^m \\ &= \exp \left(\sum_{\gamma_\sigma \in \Gamma_\sigma} \frac{w(\gamma_\sigma)}{|\text{Aut}(\gamma_\sigma)|} \right), \end{aligned} \quad (10)$$

where Γ_σ denotes the set of non-isomorphic cycle graphs. For a cycle σ of order $|\sigma| = t$ we have $\text{Aut}(\gamma_\sigma) = \langle \sigma^s \rangle$ for some $s|t$ with $|\text{Aut}(\gamma_\sigma)| = \frac{t}{s}$. Using the trace identity

$$\sum_{\gamma_\sigma, |\sigma|=t} s w(\gamma_\sigma) = \beta \text{Tr}(A^t),$$

we find

$$\begin{aligned}
\sum_{\gamma_\sigma \in \Gamma_\sigma} \frac{w(\gamma_\sigma)}{|\text{Aut}(\gamma_\sigma)|} &= \beta \sum_{t \geq 1} \frac{1}{t} \text{Tr}(A^t) \\
&= -\beta \text{Tr} \log(I - A) \\
&= -\beta \log \det(I - A).
\end{aligned}$$

Thus

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \text{perm}_\beta A(n^{\mathbf{r}}, n^{\mathbf{r}}) = \det(I - A)^{-\beta}. \quad \square$$

Let $w_1(\gamma)$ denote the weight for γ with $\beta = 1$ in (6). Define a cycle to be primitive (or rotationless) if $|\text{Aut}(\gamma_\sigma)| = 1$. For a general cycle σ with $|\text{Aut}(\gamma_\sigma)| = k$ we have $\gamma_\sigma = \gamma_\rho^k$ for a primitive cycle ρ . Let Γ_ρ denote the set of all primitive cycles. Then

$$\begin{aligned}
\sum_{\gamma_\sigma \in \Gamma_\sigma} \frac{w_1(\gamma_\sigma)}{|\text{Aut}(\gamma_\sigma)|} &= \sum_{\gamma_\rho \in \Gamma_\rho} \sum_{k \geq 1} \frac{1}{k} w_1(\gamma_\rho)^k \\
&= - \sum_{\gamma_\rho \in \Gamma_\rho} \log \det(1 - w_1(\gamma_\rho)).
\end{aligned}$$

Combining this with (10) implies [MT1]

Proposition 2.3

$$\det(I - A) = \prod_{\gamma_\rho \in \Gamma_\rho} (1 - w_1(\gamma_\rho)).$$

3 The Submatrix MMT

Our first generalization of Theorem 2.1 concerns submatrices. Consider an $(n' + n) \times (n' + n)$ matrix with block structure

$$\begin{bmatrix} B & U \\ V & A \end{bmatrix}, \tag{11}$$

where $A = (A_{ij})$ is an $n \times n$ matrix indexed by i, j , $B = (B_{i'j'})$ is an $n' \times n'$ matrix indexed by i', j' , $U = (U_{i'j})$ is an $n' \times n$ matrix and $V = (V_{ij'})$ is an $n \times n'$ matrix. For a multiset $n^{\mathbf{r}}$ of size N define the $(n' + N) \times (n' + N)$ matrix

$$\begin{bmatrix} B & U(n^{\mathbf{r}}) \\ V(n^{\mathbf{r}}) & A(n^{\mathbf{r}}, n^{\mathbf{r}}) \end{bmatrix}, \quad (12)$$

where, as before, $A(n^{\mathbf{r}}, n^{\mathbf{r}})$ denotes the $N \times N$ matrix indexed by $n^{\mathbf{r}}$, $U(n^{\mathbf{r}})$ is an $n' \times N$ matrix and $V(n^{\mathbf{r}})$ is an $N \times n'$ matrix. We then find

Theorem 3.1

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \text{perm}_{\beta} \begin{bmatrix} B & U(n^{\mathbf{r}}) \\ V(n^{\mathbf{r}}) & A(n^{\mathbf{r}}, n^{\mathbf{r}}) \end{bmatrix} = \frac{\text{perm}_{\beta} \tilde{B}}{\det(I - A)^{\beta}}, \quad (13)$$

for $n' \times n'$ matrix

$$\tilde{B} = B + U(I - A)^{-1}V,$$

where $(I - A)^{-1} = \sum_{k \geq 0} A^k$.

This result is related to Theorem 10 of [MT1] when $\beta = 1$ and Theorem 2 of [TZ1] for $\beta = -1$.

Proof. Let $\mathbf{n} = \{1, \dots, n\}$ and $\mathbf{n}' = \{1', \dots, n'\}$ and let $\mathbf{n}' \cup n^{\mathbf{r}}$ denote the multiset indexing the block matrix (12). Define a permutation graph γ_{π} with weight $w(\gamma_{\pi})$ for each $\pi \in \Sigma(\mathbf{n}' \cup n^{\mathbf{r}})$ as follows. Each vertex is labelled by an element of \mathbf{n} or \mathbf{n}' which we refer to as \mathbf{n} -vertex or \mathbf{n}' -vertex respectively. For $l = \pi(k)$ with $k, l \in \mathbf{n}' \cup n^{\mathbf{r}}$ we define an edge $e_{kl} = k \bullet \longrightarrow \bullet l$ with weight

$$w(e_{kl}) = \left[\begin{array}{cc} B & U(n^{\mathbf{r}}) \\ V(n^{\mathbf{r}}) & A(n^{\mathbf{r}}, n^{\mathbf{r}}) \end{array} \right]_{kl}.$$

Define a weight for γ_{π} by

$$w(\gamma_{\pi}) = \beta^{C(\pi)} \prod_{e_{kl} \in \gamma_{\pi}} w(e_{kl}),$$

where $C(\pi)$ is the number of cycles in π . As before, we find

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \text{perm}_{\beta} \begin{bmatrix} B & U(\mathbf{k}) \\ V(\mathbf{k}) & A(\mathbf{k}, \mathbf{k}) \end{bmatrix} = \sum_{\gamma \in \hat{\Gamma}} \frac{w(\gamma)}{|\text{Aut}(\gamma)|},$$

where $\widehat{\Gamma}$ denotes the set of non-isomorphic graphs. Each $\gamma \in \widehat{\Gamma}$ has a decomposition into cycles γ_{σ_a} which contain \mathbf{n} -vertices only and cycles $\gamma_{\sigma'_b}$ which contain at least one \mathbf{n}' -vertex:

$$\gamma = \gamma_{\sigma_1}^{m_1} \cdots \gamma_{\sigma_K}^{m_K} \gamma_{\sigma'_1} \cdots \gamma_{\sigma'_L},$$

with weight

$$w(\gamma) = \prod_a w(\gamma_{\sigma_a})^{m_a} \prod_b w(\gamma_{\sigma'_b}).$$

The set of non-isomorphic γ_{σ_a} cycle graphs labelled by \mathbf{n} is equivalent to Γ_σ introduced in the proof of Theorem 2.1. Since each \mathbf{n}' -vertex occurs exactly once in γ , each $\gamma_{\sigma'_b}$ cycle occurs at most once and has trivial automorphism group. Hence

$$|\text{Aut}(\gamma)| = \prod_a |\text{Aut}(\gamma_{\sigma_a})|^{m_a} m_a!,$$

as before. Thus the sum over weights of all graphs decomposes into the product

$$\begin{aligned} \sum_{\gamma \in \widehat{\Gamma}} \frac{w(\gamma)}{|\text{Aut}(\gamma)|} &= \sum_{\gamma_{\sigma'}} w(\gamma_{\sigma'}) \prod_{\gamma_\sigma \in \Gamma_\sigma} \sum_{m \geq 0} \frac{w(\gamma_\sigma)^m}{|\text{Aut}(\gamma_\sigma)|^m m!} \\ &= \frac{\sum_{\gamma_{\sigma'}} w(\gamma_{\sigma'})}{\det(I - A)^\beta}, \end{aligned}$$

using Theorem 2.1 and where $\gamma_{\sigma'}$ ranges over non-isomorphic cycles in $\widehat{\Gamma}$ containing at least one \mathbf{n}' -vertex.

It remains to compute $\sum_{\gamma_{\sigma'}} w(\gamma_{\sigma'})$. Let $\sigma' \in \Sigma_{n'}$ denote the permutation cycle corresponding to the cyclic sequence of \mathbf{n}' -vertices in a $\gamma_{\sigma'}$ -cycle (for arbitrary intermediate \mathbf{n} -vertices). The total edge weight coming from all subgraphs, illustrated in Fig. 2, joining two \mathbf{n}' -vertices, i' and j' , summed over all intermediate \mathbf{n} -vertices is

$$\begin{aligned} &B_{i'j'} + (UV)_{i'j'} + (UAV)_{i'j'} + (UA^2V)_{i'j'} + \dots \\ &= (B + U(I - A)^{-1}V)_{i'j'} = \widetilde{B}_{i'j'}. \end{aligned}$$

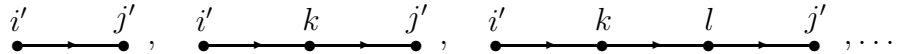


Fig. 2

Thus the total weight of all $\gamma_{\sigma'}$ cycles for a given \mathbf{n}' -vertex cycle $\sigma' = (i'_1 \dots i'_p)$ is $\beta \prod_l \tilde{B}_{i'_l \sigma'(i'_l)}$. Altogether, it follows that

$$\begin{aligned} \sum_{\gamma_{\sigma'}} w(\gamma_{\sigma'}) &= \sum_{\pi' \in \Sigma_{\mathbf{n}'}} \beta^{C(\pi')} \prod_{i'} \tilde{B}_{i' \pi'(i')} \\ &= \text{perm}_{\beta} \tilde{B}. \quad \square \end{aligned}$$

Lemma 3.2 *For $\beta = -1$ Theorem 3.1 implies*

$$\sum_{r_i \in \{0,1\}} \det \begin{bmatrix} B & U(n^{\mathbf{r}}) \\ V(n^{\mathbf{r}}) & A(n^{\mathbf{r}}, n^{\mathbf{r}}) \end{bmatrix} = \det \begin{bmatrix} B & -U \\ -V & I + A \end{bmatrix}.$$

Proof. For $\beta = -1$ the right hand side of (13) gives

$$\begin{aligned} \text{perm}_{-1} \tilde{B} \det(I - A) &= (-1)^{n'} \det(B + U(I - A)^{-1}V) \det(I - A) \\ &= \det \begin{bmatrix} -B & U \\ V & I - A \end{bmatrix}, \end{aligned}$$

by means of the matrix identity

$$\begin{bmatrix} -B & U \\ V & I - A \end{bmatrix} = \begin{bmatrix} -I' & U(I - A)^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} B + U(I - A)^{-1}V & 0 \\ V & I \end{bmatrix} \begin{bmatrix} I' & 0 \\ 0 & I - A \end{bmatrix},$$

where I and I' are respectively $n \times n$ and $n' \times n'$ identity matrices. The result follows on replacing A, B, U, V by $-A, -B, -U, -V$. \square

4 The Partial Permutation MMT

The next generalization of Theorem 2.1 is concerned with replacing permutations by partial permutations with a suitable generalization of the notions of permanent and β -extended permanent. Let Ψ denote the set of partial permutations of the set $\{1, \dots, n\}$ i.e. injective partial mappings from $\{1, \dots, n\}$ to itself. For $\psi \in \Psi$ we let $\text{dom } \psi$ and $\text{im } \psi$ denote the domain and image respectively and let π_{ψ} denote the (possibly empty) permutation of $\text{dom } \psi \cap \text{im } \psi$ determined by ψ .

We introduce the Partial Permanent of an $n \times n$ matrix $A = (A_{ij})$ indexed by $i, j \in \{1, \dots, n\}$ as follows

$$\text{pperm}A = \sum_{\psi \in \Psi} \prod_{i \in \text{dom } \psi} A_{i\psi(i)}, \quad (14)$$

with unit contribution for the empty map. Let $\theta = (\theta_i), \phi = (\phi_i)$ be n -vectors and define the (β, θ, ϕ) -extended Partial Permanent by

$$\text{pperm}_{\beta\theta\phi}A = \sum_{\psi \in \Psi} \beta^{C(\pi_\psi)} \prod_{i \in \text{dom } \psi} A_{i\psi(i)} \prod_{j \notin \text{im } \psi} \theta_j \prod_{k \notin \text{dom } \psi} \phi_k, \quad (15)$$

where $C(\pi_\psi)$ is the number of cycles in π_ψ e.g.

$$\begin{aligned} \text{pperm}_{\beta\theta\phi} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} &= \theta_1\phi_1\theta_2\phi_2 + \beta(A_{11}\theta_2\phi_2 + A_{22}\theta_1\phi_1) \\ &\quad + A_{12}\theta_1\phi_2 + A_{21}\theta_2\phi_1 + \beta^2 A_{11}A_{22} + \beta A_{12}A_{21}. \end{aligned}$$

A recent application of an extended partial permanent appears in [HT].

Let $A(n^{\mathbf{r}}, n^{\mathbf{r}})$ denote the $N \times N$ matrix indexed by a multiset $n^{\mathbf{r}}$ as before. We also let $\text{pperm}_{\beta\theta\phi}A(n^{\mathbf{r}}, n^{\mathbf{r}})$ denote the corresponding partial permanent with N -vectors $(\theta_{1_1}, \dots, \theta_{n_{r_n}})$ and $(\phi_{1_1}, \dots, \phi_{n_{r_n}})$. We then find

Theorem 4.1

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \text{pperm}_{\beta\theta\phi}A(n^{\mathbf{r}}, n^{\mathbf{r}}) = \frac{e^{\theta(I-A)^{-1}\phi^T}}{\det(I-A)^\beta}, \quad (16)$$

where ϕ^T denotes the transpose of the row vector ϕ .

This result is related to Theorem 11 of [MT1] for $\beta = 1$.

Proof. Let $\Psi(n^{\mathbf{r}})$ denote the partial permutations of $n^{\mathbf{r}}$. Define a partial permutation graph γ_ψ labelled by $\{1, \dots, n\}$ for each $\psi \in \Psi(\mathbf{k})$ with edges

$$e_{ij} = i \bullet \longrightarrow \bullet j,$$

for $j = \psi(i)$ with $i \in \text{dom } \psi$ and $j \in \text{im } \psi$. Let v_i denote the vertex of γ_ψ with label i . If $i \notin \text{dom } \psi$ then either $\deg v_i = 0$ or $\deg v_i = \text{indeg } v_i = 1$ whereas if $i \notin \text{im } \psi$ then either $\deg v_i = 0$ or $\deg v_i = \text{outdeg } v_i = 1$. In all other cases

permutation $\psi = \begin{pmatrix} 1_1 & 1_2 & 1_3 & 2_1 & 2_2 & 4_1 \\ 2_1 & & 4_1 & 1_2 & & 1_3 \end{pmatrix}$ then γ_ψ is shown in Fig. 3. In this case $\text{dom } \psi = \{1_1, 1_3, 2_1, 4_1\}$ and $\text{im } \psi = \{1_2, 1_3, 2_1, 4_1\}$ and $\pi_\psi = (1_3 4_1)$.

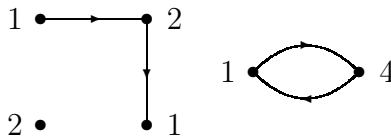


Fig. 3

Define an edge weight as before by $w(e_{ij}) = A_{ij}$ and introduce a vertex weight

$$w(v_k) = \begin{cases} 1, & \deg v_k = 2, \\ \theta_k, & \deg v_k = \text{outdeg } v_k = 1, \\ \phi_k, & \deg v_k = \text{indeg } v_k = 1, \\ \theta_k \phi_k, & \deg v_k = 0. \end{cases}$$

The weight of a graph γ_ψ is defined by

$$w(\gamma_\psi) = \beta^{C(\pi_\psi)} \prod_{e_{ij}} w(e_{ij}) \prod_{v_k} w(v_k),$$

where $C(\pi_\psi)$ is the number of cycles in π_ψ . The weight is multiplicative with respect to the cycle and necklace decomposition. We find again that

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \text{pperm}_{\beta\theta\phi} A(n^{\mathbf{r}}, n^{\mathbf{r}}) = \sum_{\gamma \in \tilde{\Gamma}} \frac{w(\gamma)}{|\text{Aut}(\gamma)|},$$

where $\tilde{\Gamma}$ denotes the set of non-isomorphic graphs. Each $\gamma \in \tilde{\Gamma}$ has a decomposition into connected cycle graphs γ_{σ_a} and open necklaces ν_b :

$$\gamma = \nu_1^{l_1} \dots \nu_L^{l_1} \gamma_{\sigma_1}^{m_1} \dots \gamma_{\sigma_K}^{m_K},$$

with weight

$$w(\gamma) = \prod_b w(\nu_b)^{l_b} \cdot \prod_a w(\gamma_{\sigma_a})^{m_a}.$$

Each necklace has trivial automorphism group but can have multiple occurrences. Hence we find that

$$|\text{Aut}(\gamma)| = \prod_b l_b! \cdot \prod_a |\text{Aut}(\gamma_{\sigma_a})|^{m_a} m_a!.$$

Thus the sum over weights of all graphs decomposes into the product

$$\begin{aligned} \sum_{\gamma \in \tilde{\Gamma}} \frac{w(\gamma)}{|\text{Aut}(\gamma)|} &= \prod_{\nu \in \Gamma_\nu} \sum_{l \geq 0} \frac{w(\nu)^l}{l!} \cdot \prod_{\gamma_\sigma \in \Gamma_\sigma} \sum_{m \geq 0} \frac{w(\gamma_\sigma)^m}{|\text{Aut}(\gamma_\sigma)|^m m!} \\ &= \exp \left(\sum_{\nu \in \Gamma_\nu} w(\nu) \right) \frac{1}{\det(I - A)^\beta}, \end{aligned}$$

where Γ_ν denotes the set of non-isomorphic open necklaces and using Theorem 2.1 again. Finally, the sum over the weights of connected necklaces, such as depicted in Fig. 4, is

$$\begin{aligned} \sum_{\nu \in \Gamma_\nu} w(\nu) &= \theta \phi^T + \theta A \phi^T + \theta A^2 \phi^T + \dots \\ &= \theta (I - A)^{-1} \phi^T. \quad \square \end{aligned}$$

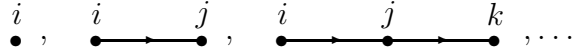


Fig. 4

Example. Consider $n = 1$ with $A = z$ and $\theta_1 = \phi_1 = \sqrt{\alpha z}$. Then we find

$$\text{pperm}_{\beta\theta\phi} A(1^r, 1^r) = p_r(\alpha, \beta) z^r,$$

where $p_r(\alpha, \beta) = \sum_{s,t} p_{rst} \alpha^s \beta^t$ is the generating polynomial for p_{rst} the number of graphs with r identically labelled vertices, s open necklaces and t cycles. Theorem 4.1 provides the exponential generating function for $p_r(\alpha, \beta)$ [HT]

$$\sum_{r \geq 0} \frac{p_r(\alpha, \beta)}{r!} z^r = \frac{\exp\left(\frac{\alpha z}{1-z}\right)}{(1-z)^\beta}.$$

5 The Submatrix Partial Permutation MMT

We can combine the two generalizations above into one theorem concerning partial permutations of submatrices of the $(n' + n) \times (n' + n)$ block matrix (11). Let $\theta' = (\theta'_{i'})$ and $\phi' = (\phi'_{i'})$ be n' -vectors and $\theta = (\theta_i)$ and $\phi = (\phi_i)$ be n -vectors. For a multiset $n^{\mathbf{r}}$ of size N and block matrix (12) labelled by $\mathbf{n}' = \{1', \dots, n'\}$ and $n^{\mathbf{r}}$, we let $\text{pperm}_{\beta\theta\phi} \begin{bmatrix} B & U(n^{\mathbf{r}}) \\ V(n^{\mathbf{r}}) & A(n^{\mathbf{r}}, n^{\mathbf{r}}) \end{bmatrix}$ denote the (β, θ, ϕ) -extended partial permanent with $(n' + N)$ -vectors $(\theta'_{1'}, \dots, \theta'_{n'}, \theta_{1_1}, \dots, \theta_{n_{r_n}})$ and $(\phi'_{1'}, \dots, \phi'_{n'}, \phi_{1_1}, \dots, \phi_{n_{r_n}})$ respectively. We then find

Theorem 5.1

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \text{pperm}_{\beta\theta\phi} \begin{bmatrix} B & U(n^{\mathbf{r}}) \\ V(n^{\mathbf{r}}) & A(n^{\mathbf{r}}, n^{\mathbf{r}}) \end{bmatrix} = \frac{e^{\theta(I-A)^{-1}\phi^T} \cdot \text{pperm}_{\beta\tilde{\theta}\tilde{\phi}} \tilde{B}}{\det(I-A)^\beta}, \quad (17)$$

for

$$\begin{aligned} \tilde{B} &= B + U(I-A)^{-1}V, \\ \tilde{\theta} &= \theta' + \theta(I-A)^{-1}V, \\ \tilde{\phi}^T &= \phi'^T + U(I-A)^{-1}\phi^T. \end{aligned}$$

This result is related to Theorem 13 of [MT1] for $\beta = 1$.

Proof. We sketch the proof since it runs along very similar lines to the preceding ones. Define a partial permutation graph γ_ψ for each partial permutation ψ of $\mathbf{n}' \cup n^{\mathbf{r}}$. In this case, the connected subgraphs consist of cycle graphs Γ_σ and open necklaces Γ_ν containing only \mathbf{n} -vertices, and cycles and open necklaces containing at least one \mathbf{n}' -vertex. Define a graph weight $w(\gamma_\psi)$ as a product of edge weights, vertex weights and cycle factors as before. This results in

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \text{pperm}_{\beta\theta\phi} \begin{bmatrix} B & U(n^{\mathbf{r}}) \\ V(n^{\mathbf{r}}) & A(n^{\mathbf{r}}, n^{\mathbf{r}}) \end{bmatrix} = \frac{e^{\theta(I-A)^{-1}\phi^T}}{\det(I-A)^\beta} \sum_{\gamma' \in \Gamma'} w(\gamma'),$$

where the sum is over all graphs Γ' containing at least one \mathbf{n}' -vertex. The remaining terms arise as before.

Each $\gamma' \in \Gamma'$ canonically determines a partial permutation $\psi' \in \Psi(\mathbf{n}')$ described by the corresponding ordered sequences of \mathbf{n}' -vertices (for any intermediate \mathbf{n} -vertices). As before, the total edge weight coming from all

subgraphs joining two \mathbf{n}' -vertices i' and j' with intermediate \mathbf{n} -vertices is $\tilde{B}_{i'j'}$. The total weight arising from the subgraphs of all necklaces joining \mathbf{n} -vertices to an \mathbf{n}' -vertex i' with intermediate \mathbf{n} -vertices as depicted in Fig. 5 is

$$\begin{aligned} & \theta'_{i'} + (\theta V)_{i'} + (\theta AV)_{i'} + \dots \\ &= (\theta' + \theta(I - A)^{-1}V)_{i'} = \tilde{\theta}_{i'}. \end{aligned}$$

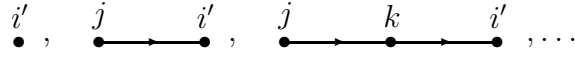


Fig. 5

Likewise, the total weight arising from all subgraphs joining an \mathbf{n}' -vertex j' to \mathbf{n} -vertices with intermediate \mathbf{n} -vertices is $\tilde{\phi}_{j'}$. Combining these results we find that

$$\sum_{\gamma' \in \Gamma'} w(\gamma') = \text{pperm}_{\beta \tilde{\theta} \tilde{\phi}} \tilde{B}. \quad \square$$

6 The Derangement MMT

Let $\Delta_n \subset \Sigma_n$ denote the derangements of the set $\{1, \dots, n\}$ i.e. each $\pi \in \Delta_n$ contains no cycles of length 1. We introduce the β -extended Deranged Permanent of an $n \times n$ matrix A by

$$\text{dperm}_{\beta} A = \sum_{\pi \in \Delta_n} \beta^{C(\pi)} \prod_i A_{i\pi(i)}. \quad (18)$$

Using the same multiset notation as before we find

Theorem 6.1

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \text{dperm}_{\beta} A(n^{\mathbf{r}}, n^{\mathbf{r}}) = \frac{e^{-\beta \text{Tr} A}}{\det(I - A)^{\beta}}. \quad (19)$$

Proof. Following the proof of Theorem 2.1 we find

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \text{dperm}_{\beta} A(n^{\mathbf{r}}, n^{\mathbf{r}}) = \exp \left(\sum_{\gamma_{\sigma} \in \Gamma_{\sigma}, |\sigma| \geq 2} \frac{w(\gamma_{\sigma})}{|\text{Aut}(\gamma_{\sigma})|} \right),$$

where cycles of length one are excluded. Using

$$\begin{aligned} \sum_{\gamma_{\sigma} \in \Gamma_{\sigma}, |\sigma| \geq 2} \frac{w(\gamma_{\sigma})}{|\text{Aut}(\gamma_{\sigma})|} &= \beta \sum_{s \geq 1} \frac{1}{s} \text{Tr}(A^s) - \beta \text{Tr} A \\ &= -\beta \text{Tr} \log(I - A) - \beta \text{Tr} A, \end{aligned}$$

the result follows. \square

Example. Consider $n = 1$ with $A = z$. Then for multisets $\{1^r\}$ we find

$$\text{dperm}_{\beta} A(1^r, 1^r) = d_r(\beta) z^r,$$

where $d_r(\beta) = \sum_s d_{rs} \beta^s$ is the generating polynomial for d_{rs} the number of derangements of r labels with s cycles. From Theorem 6.1 the exponential generating function for $d_r(\beta)$ is [HT]

$$\sum_{r \geq 0} \frac{1}{r!} z^r d_r(\beta) = \left(\frac{e^{-z}}{1 - z} \right)^{\beta}.$$

Finally, we can further generalize Theorem 6.1 to deranged permanents of submatrices as in Theorem 3.1. Using the notation of (11) and (12) we find using similar techniques that

Theorem 6.2

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \text{dperm}_{\beta} \begin{bmatrix} B & U(n^{\mathbf{r}}) \\ V(n^{\mathbf{r}}) & A(n^{\mathbf{r}}, n^{\mathbf{r}}) \end{bmatrix} = \frac{e^{-\beta \text{Tr} A} \cdot \text{perm}_{\beta} \hat{B}}{\det(I - A)^{\beta}}, \quad (20)$$

for $n' \times n'$ matrix

$$\hat{B} = B - \text{diag } B + U(I - A)^{-1} V,$$

where $\text{diag } B_{i'j'} = B_{i'i'} \delta_{i'j'}$. \square

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